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Theory of a Probe in a Strong Magnetic Field^{*}

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ABSTRACT

A kinetic approach is used to develop a theory of electrostatic probes in a fully ionized plasma in the presence of a magnetic field. A consistent asymptotic expansion is obtained assuming that the electron Larmor radius is small compared to the radius of the probe. The order of magnitude of neglected terms is given. It is found that the electric potential within the tube of force defined by the cross section of the probe decays non-monotonically from the probe; this bump disappears at a certain probe voltage and the theory is valid up to this voltage. The transition region, which extends beyond plasma potential, is not exponential. The possible saturation of the electron current is discussed. Restricted numerical results are given; they seem to be useful for weaker magnetic fields down to the zero-field limit. Extensions of the theory are considered.

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I. INTRODUCTION

Since Langmuir's work,¹ probe theory has been extensively developed for the case of zero magnetic field ($B = 0$). The theory is practically complete²⁻⁴ in the dilute limit when $\lambda_D \ll R \ll \lambda$ (λ_D , R , and λ being the Debye length, probe radius, and mean free path, respectively). Extensive computed results are available in Ref. 5, including values of $\lambda_D/R > 0(1)$. The continuum case $\lambda \ll \lambda_D \ll R$ was successfully treated by Su and Lam⁶ and Cohen.⁷ The more difficult problem of an intermediate λ has been approached rather crudely,⁸ although lately some improvements have been obtained.^{9,10}

When a magnetic field is present no reliable results exist. The probe characteristic or current-voltage ($I-V_p$) diagram has less definite features: roughly, a certain decrease in the current and a blurring of the plasma potential kink. Papers on the subject are crude and sparse. Three basic difficulties are the cause of this. First, the presence of \underline{B} makes the space anisotropic. Second, for large B , collisions will come into play; unless λ is the smallest relevant length, the character of the equations will change substantially in space. Finally, for fully ionized plasmas, transport coefficients are spatially dependent because the density experiences large changes; in fact, for either $B = 0$ or $B \neq 0$, only one paper¹¹ is known to the author where Coulomb collisions are considered, although as a weak effect.

Spivak and Reichrudel¹² studied the case of a weak field. As in later papers,^{13,14} where plane probes parallel to \underline{B} were assumed, an ill-defined sheath edge is introduced where the density cannot be expected to be that of

the unperturbed plasma. Moreover, the effect of the flux along \underline{B} is not considered at all.

Bertotti¹⁵ treated the case of a probe perpendicular to \underline{B} and averaged magnitudes over the cross section. An unspecified diffusion process was assumed and in this way a phenomenological integro-differential equation was found and integrated numerically. The results are in clear contradiction to all experimental evidence.

Bohm¹⁶ established a balancing between fluxes along and across \underline{B} . Besides the known depletion of the plasma he found the results to be insensitive to the shape of the probe along \underline{B} . However, several defects should be pointed out. First, the diffusion equation is assumed to be valid up to a vaguely-defined surface: no proper matching is made, the results are quantitatively unreliable, and there is no way of knowing the error of the approximations or of improving the treatment. Second, the value of the current, I , does not depend on V_p and so it is useless (the curve certainly is not flat around space potential, $V_p = 0$). The ion-electron temperature ratio was considered negligible. A correction by Sugawara¹⁷ is wrong; the involved I - V_p dependence produced by an overshooting of the potential is missed. Moreover, in both papers Laplace's equation was solved incorrectly: the density was assumed constant on the above-indicated surface. They were concerned with weakly ionized gases.

Our treatment is of an asymptotic nature and quiescent fully ionized plasmas are considered. We shall formulate our problem and assumptions

in Sec. II. In Sec. III a basic physical picture is obtained, and the equations are properly solved in Sec. IV. In Sec. V we present the actual study of the $I-V_p$ diagram. Extensions of the theory and discussions of results are given in the last section.

II. STATEMENT OF THE PROBLEM

Two shapes of probes will be considered simultaneously: a thin strip of width $2R$ and infinitely long, and a disc of radius R , both perpendicular to \underline{B} . They give rise to two-dimensional and axisymmetric problems, respectively. The strip will be found to disturb entirely the plasma, as is frequent whenever diffusion is important. The probe is normally cold and acts as a sink of particles; it will be assumed perfectly absorbing. This condition is easily relaxed (see Sec. VI).

The equations to be studied are

$$\underline{w} \cdot \frac{\partial F^j}{\partial \underline{x}} - \frac{q_j}{m_j} \frac{\partial V}{\partial \underline{x}} \cdot \frac{\partial F^j}{\partial \underline{w}} + \frac{q_j}{m_j} \underline{w} \times \underline{B} \cdot \frac{\partial F^j}{\partial \underline{w}} = \sum_k C(F^j, F^k) \quad (1)$$

($j = e, i$ for electrons and ions, respectively)

$$\nabla^2 V = 4\pi e (N^e - Z_i N^i) \quad (2)$$

$$N^j = \int F^j d\underline{w}.$$

The boundary conditions are

$$V = V_p \quad \text{on the probe}$$

$$V = 0 \quad \text{at infinity}$$

and the F^j 's given at infinity; in particular, there,

$$N_e = Z_i N_i = N_\infty$$

$$T_e = T_{e\infty} = T_\infty$$

$$T_i = T_{i\infty} = \beta Z_i T_\infty ,$$

β being $T_{i\infty}/T_{e\infty} Z_i$ and $Z_i e$ the charge of the ions. Space (or plasma) potential is taken as origin of voltages.

We now define non-dimensional variables:

$$f^e = \frac{F^e U_e^3}{N_\infty} , \quad f^i = \frac{F^i U_i^3}{N_\infty / Z_i}$$

$$\chi = \frac{eV}{kT_\infty} , \quad \underline{v} = \frac{\underline{w}}{U_j} \quad (j = e, i)$$

where $U_j^2 m_j = kT_{j\infty}$. Then Eqs. (1), (2) become explicitly (choosing the z axis along \underline{B})

$$\left[v_z \frac{\partial}{\partial z} + v_\xi \frac{\partial}{\partial \xi} + \frac{\partial \chi}{\partial z} \frac{\partial}{\partial v_z} + \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial v_\xi} - \left(\frac{s v}{\xi} - \frac{1}{\ell_e} \right) \left(v_\xi \frac{\partial}{\partial v_\eta} - v_\eta \frac{\partial}{\partial v_\xi} \right) \right] f^e =$$

$$= \lambda^{-1} [c(f^e, f^e) + c(f^e, f^i)] \quad (3)$$

$$\left[v_z \frac{\partial}{\partial z} + v_\xi \frac{\partial}{\partial \xi} - \frac{1}{\beta} \frac{\partial \chi}{\partial z} \frac{\partial}{\partial v_z} - \frac{1}{\beta} \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial v_\xi} - \left(\frac{s v}{\xi} + \frac{1}{\ell_i} \right) \left(v_\xi \frac{\partial}{\partial v_\eta} - v_\eta \frac{\partial}{\partial v_\xi} \right) \right] f^i =$$

$$= \lambda^{-1} [c(f^i, f^i) + c(f^i, f^e)] \quad (4)$$

$$\lambda_D^2 \left[\frac{\partial^2}{\partial z^2} + \xi^{-s} \frac{\partial}{\partial \xi} \xi^s \frac{\partial}{\partial \xi} \right] \chi = n^e - n^i \quad (5)$$

where

$$\begin{aligned} \ell_j &= \frac{m_j U_j c}{|q_j| B}, \quad n^j = \int f^j dv \\ \lambda_D &= \left(\frac{k \cdot T_\infty}{4\pi N_\infty e^2} \right)^{1/2} \quad \lambda = \frac{(kT_\infty)^2}{2\pi N_\infty e^4 \ln \Lambda_\infty} \\ \Lambda_\infty &= \frac{3kT_\infty}{e^2} \left[\frac{kT_\infty}{4\pi N_\infty e^2 (1+\beta^{-1})} \right]^{1/2} \end{aligned} \quad (6)$$

$s = 0$ for the strip and 1 for the disc. In this case ξ is the radial distance in cylindrical coordinates; for $s = 0$ the $z - \xi$ plane is perpendicular to the strip.

The definition of λ is somewhat arbitrary but, of course, the solution will not depend on it. The point is that the c 's are now non-dimensional quantities of order unity for $|v| = O(1)$, as are the terms on the left-hand sides of (3)-(4) if exclusion is made of spatial derivatives or length factors. The identities

$$\lambda^{-1} \sum_k c(f^j, f^k) = \frac{U_j^2}{N_{j\infty}} \sum_k C(F^j, F^k)$$

define the c 's. It is important to observe that we can write

$$c(f^e, f^i) = n^i P(f^e) + \mu P_\mu(f^e, f^i) \quad (7)$$

where $P(f^e) = 0$ if f^e is isotropic and $\mu P_\mu = O(\mu)$, $\mu \sim (m^e/m_i)^{1/2}$.

The C 's are collision operators of the Fokker-Planck or Balescu-Lenard type if λ_D is smaller than any gradient length or the length over which any field produces a non-negligible effect on the motion of the particles (for B ,

the Larmor radius). In other cases the proper non-local operators will be assumed; under the weak restriction that e^2/kT_∞ should be the smallest length present, the c 's are still $O(1)$ with λ given by (6).

The boundary conditions are

$$\chi = \chi_p = eV_p/kT_\infty \quad \text{on the probe}$$

$$\chi = 0 \quad \text{at infinity}$$

$$f^e, f^i \text{ given at infinity where } n^e = n^i = 1.$$

Five characteristic lengths appear explicitly in our equations: λ_D , λ , R , ℓ_e , ℓ_i . We can define four non-dimensional parameters

$$\mu = \ell_e/\ell_i < O(1) \quad (\text{i.e., } \mu \ll 1)$$

$$\sigma = \ell_e/R < O(1)$$

$$\varepsilon = \lambda_D/R < O(1)$$

$$z = R/\lambda \leq O(1).$$

The first is a natural small parameter of our plasma [β is arbitrary but $O(1)$]. $\sigma < O(1)$ will be considered as our definition of a "strong" magnetic field; in Sec. VI the use of the present results for weaker fields down to the limit $B \rightarrow 0$ ($\sigma \rightarrow \infty$) will be discussed. We require also the exclusion of the case $\sigma/\mu < O(1)$.

The condition $\varepsilon < O(1)$ is frequently satisfied in actual plasmas; the complications arising from the relaxation of this condition will be considered.

In the last restriction we exclude the case $\lambda \ll R$ but not $\lambda < R$.

Furthermore,

$$\varepsilon\tau = \lambda_D/\lambda = O(\Lambda_\infty^{-1} \ln \Lambda_\infty) < O(1).$$

Thus if $\tau = O(1)$, $\varepsilon < O(1)$; conditions on ε and τ are not entirely different.

It is worth pointing out here that writing Eqs. (3) and (4) excludes the possibility of anomalous diffusion due to fluctuating fields. The consequences of this strong restriction will come out clearly from our analysis. It can be guessed at present that a certain degree of smoothness in the plasma is to be required. The allowance for $\tau = O(1)$ makes our treatment useful for a relatively large range of cases. In itself, the interaction of body, plasma, and magnetic field is of interest.

III. THE PERTURBATION IN SPACE

Consider a cylindrical probe in the absence of a magnetic field. The directed flux toward the probe decays as r^{-1} where r is the distance to the axis of the cylinder. This results in an immediate, obvious first integral of the continuity equation. If the probe is small enough the spreading makes negligible all effects at distances still "microscopic."

However, when a magnetic field is present, no such integral is available. The continuity equation is not eliminated and clearly will be of foremost importance. For large enough B , electrons are inhibited from flowing across field lines so that the flux along \underline{B} will vary slowly and transport coefficients will come into play. Moreover, in the plane of the probe ($z = 0$) and outside its area, the parallel flux is zero by symmetry but it is not so inside. Therefore strong gradients will appear on the boundary of the "shadow," the tube of force impinging on the probe. If we consider any of

the two half-spaces produced by the plane of the probe the problem is equivalent to that of a plasma bounded by an infinite wall where a discontinuity on the reflection coefficient occurs over a small circle.

To obtain an initial picture of the situation we apply briefly a multiple scales method in this section. First, we define

$$\left. \begin{aligned} D_z &= v_z \frac{\partial}{\partial z} + \frac{\partial \chi}{\partial z} \frac{\partial}{\partial v_z} \\ D_\xi &= v_\perp \frac{\partial}{\partial \xi} + \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial v_\perp} \end{aligned} \right\} \quad (8)$$

$$v_\xi = v_\perp \cos \phi, \quad v_\eta = v_\perp \sin \phi$$

so that

$$v_\xi \frac{\partial}{\partial v_\eta} - v_\eta \frac{\partial}{\partial v_\xi} = \frac{\partial}{\partial \phi}.$$

Then, multiplying Eqs. (3) and (4) by ℓ_e and rearranging terms we get

$$\begin{aligned} \frac{\partial f^e}{\partial \phi} + \ell_e D_z f^e + \ell_e \cos \phi D_\xi f^e - \ell_e \sin \phi \left(\frac{v_\perp}{\xi} + \frac{1}{v_\perp} \frac{\partial \chi}{\partial \xi} \right) \frac{\partial f^e}{\partial \phi} \\ = \sigma \tau [c(f^e, f^e) + n^i P(f^e) + \mu P_\mu(f^e, f^i)] \end{aligned} \quad (9)$$

$$\begin{aligned} \ell_e D_z^i f^i + \ell_e \cos \phi D_\xi^i f^i - \mu \frac{\partial f^i}{\partial \phi} - \ell_e \sin \phi \left(\frac{v_\perp}{\xi} - \frac{\beta^{-1}}{v_\perp} \frac{\partial \chi}{\partial \xi} \right) \frac{\partial f^i}{\partial \phi} \\ = \sigma \tau [c(f^i, f^i) + c(f^i, f^e)]. \end{aligned} \quad (10)$$

(For D_z^i, D_ξ^i the factor $(-\beta)^{-1}$ multiplies the second terms in the operators given in Eq. (8).)

We define next a set of non-dimensional variables

$$z_k = \frac{z}{L_k^z}$$

where the L_k^z 's are unknown combinations of the five characteristic lengths of our problem. We assume that from an equation such as

$$\frac{\partial A}{\partial z_k} = E \quad (11a)$$

we can obtain

$$\lim_{z_k \rightarrow \infty} E = 0. \quad (11b)$$

This results from requiring boundedness for A and non-oscillatory behavior for E.

We define also $\xi_r = \xi/R$. Although one should allow for strong gradients around $\xi = R$ and therefore define $\xi_k = (\xi - R)/L_k^\xi$ where $L_k^\xi/R \rightarrow 0$ as $(\mu, \sigma, \varepsilon) \rightarrow 0$, in this limit

$$\frac{\xi_k}{\xi_r} = (1 - \frac{R}{\xi}) \frac{R}{L_k^\xi} \rightarrow \infty$$

for any fixed $\xi \neq R$. Thus the region over which ξ_k is finite or $\partial/\partial \xi_k \neq 0$ [see Eqs. (11a, b)] shrinks to zero. Its importance will be estimated later.

Now we make the expansions

$$f^j = \sum_m f_m^j, \quad \chi = \sum_m \chi_m$$

and retain terms in (9), (10), and (5) of dominant order. It is possible to show that the smallest length L_o^z cannot be shorter than λ_D . In effect, then, $(\lambda_D/L_o^z)^2 \gg 1$ and from (5),

$$\frac{\partial^2 \chi_o}{\partial z_o^2} = 0 \quad \text{or} \quad \chi_o = az_o + b$$

but from $\lim_{z_o \rightarrow \infty} \partial/\partial z_o = 0$ we get $a = 0$. Consequently from Eq. (10),

$\partial f_o^i / \partial z_o = 0$ and from (9),

$$f_o^e = f_o^e \left(\phi - \frac{L_o^z}{\ell_e} \frac{z_o}{v_z} \right).$$

But this function cannot be periodic in ϕ if $\partial f_o^e / \partial z_o \rightarrow 0$ as $z_o \rightarrow \infty$. Thus

$\partial f_o^e / \partial \phi = \partial f_o^e / \partial z_o = 0$. This can be seen in another way: the electrons

enter the z_o layer with a distribution function symmetric around the z axis

(to zero order), because for large z_o , $\partial f_o^e / \partial z_o = 0$. This cannot be altered

by the spiraling around field lines.

Therefore, $L_o^z = \lambda_D$. We get

$$\begin{aligned} \frac{\partial^2 \chi_o}{\partial z_o^2} &= n_o^e - n_o^i \\ \frac{\partial f_o^e}{\partial \phi} &= 0, \quad \left(v_z \frac{\partial}{\partial z_o} + \frac{\ell_e}{\lambda_D} \frac{\partial \chi_o}{\partial z_o} \frac{\partial}{\partial v_z} \right) f_o^e = 0 \end{aligned} \quad (12a, b)$$

$$\left(v_z \frac{\partial}{\partial z_o} - \frac{\ell_e}{\lambda_D} \frac{1}{\beta} \frac{\partial \chi_o}{\partial z_o} \frac{\partial}{\partial v_z} \right) f_o^i = 0$$

and also $\frac{\partial}{\partial z_o} j_o^e = 0$

where $j^{ez} = \int f^e v_z d\mathbf{v} = n^e u^{ez}$.

If we let $z_o \rightarrow \infty$, (12a) remains valid but (12b) vanishes identically. We take terms of next order in Eq. (9),

$$\begin{aligned} \frac{\partial f_1^e}{\partial \phi} + \frac{\ell_e}{L_1^z} D_{z_1} f_o^e + \frac{\ell_e}{R} \cos \phi D_{\xi_r} f_o^e \\ = \sigma \tau [c(f_o^e, f_o^e) + n_o^i P(f_o^e)] . \end{aligned}$$

This equation is of the type

$$\frac{\partial f_1^e}{\partial \phi} = A(\phi) + C$$

and from the requirement of periodicity of f_1^e in ϕ :

$$C = 0, \quad \frac{\partial f_1^e}{\partial \phi} = A(\phi).$$

Therefore,

$$\frac{\ell_e}{L_1^z} = \sigma \tau \quad \text{or} \quad L_1^z = \lambda$$

$$D_{z_1} f_o^e = c(f_o^e, f_o^e) + n_o^i P(f_o^e) \quad (1)$$

$$\delta_1 f_1^e = -\sigma \sin \phi D_{\xi_r} f_o^e + \delta_{1c} f_{1c}^e$$

where the index c means constant in ϕ . Unless otherwise stated, χ_o subst for χ in Eq. (8).

f_1^e gives an η flux but no ξ flux. Again we obtain

$$\frac{\partial}{\partial z_1} j_o^{ez} = 0, \quad \frac{\partial}{\partial z_o} j_1^{ez} = 0.$$

If we now let $z_1 \rightarrow \infty$, Eq. (12) becomes

$$c(f_o^e, f_o^e) + n_o^i P(f_o^e) = 0. \quad (1)$$

Not only is an isotropic Maxwellian distribution a solution to Eq. (14) but it is the unique solution as proved in App. B. Thus $T_{e||} = T_{e\perp}$. Moreover, $T_e = T_{e\infty}$ because the electric field is small (collisions are dominant) and heating can be neglected to zero order. Thus

$$\lim_{z \rightarrow \infty} f_o^e = n_o^e f_M = n_o^e \frac{e^{-v^2/2}}{(2\pi)^{3/2}}.$$

It should be emphasized that the decomposition of Eq. (7) is valid if the average ion velocity is much smaller than that of electrons. Therefore, if the ions had acquired an average z velocity of order $(kT_e/m_e)^{1/2}$ the above conclusion would be wrong; in particular, $u_o^{ez} = 0$ would not follow. However, this would require an energy gain for the ions of order

$$m_i \left[\left(\frac{kT_e}{m_e} \right)^{1/2} \right]^2 \approx kT_e \mu^{-2}.$$

There is no source of such a large energy. In the case of Bohm's criterion¹⁶ for ion collection, an energy gain is necessary of order

$$m_i \left[\left(\frac{kT_e}{m_i} \right)^{1/2} \right]^2 = kT_e$$

and moreover it applies when $T_i \ll T_e$ and χ_p is large enough so that all electrons are repelled. Then a fraction of V_p ,

$$\frac{1}{\chi_p} V_p = kT_e,$$

accelerates the ions outside the sheath.

From Eq. (14) there results

$$\lim_{z \rightarrow \infty} j^{ez} < O(1)$$

and $j^{ez}(z=0) < O(1)$ since j^{ez} is conserved in both z_0 and z_1 to zero order. But on the probe all particles travel toward it; thus $n^e(z=0) < O(1)$ and also $n^e(z_1 \rightarrow \infty)/n^e(z=0) < O(1)$. Now the z -momentum equation in z_0 and z_1 is

$$0 = -\frac{\partial}{\partial z_k} n_o^e E_o^{ez} + n_o^e \frac{\partial \chi_o}{\partial z_k}, \quad k = 0, 1.$$

(Because $j^{ez} < O(1)$, the collision term does not contribute to this equation for $k = 1$.) Therefore if $E_o^{ez} \equiv (n_o^e)^{-1} \int f_o^e (v_z - u_z^{eo})^2 d\vec{v}$ is order unity everywhere, $\chi_o > O(1)$; in particular it is so in the limit $z_1 \rightarrow \infty$.

Hence $n^e < O(1)$ in these layers and the results $L_o^z = \lambda_D$, $L_1^z = \lambda$ based on $n^e = O(1)$ are not true (the local values for λ_D and λ have to be used but they are unknown).

The importance of χ at these large distances from the probe follows also from the analysis of the following scale; so does the order of magnitude of n^e and χ in the limit $z_1 \rightarrow \infty$. This and the detailed solution of the problem will be made in the next section. Here we consider briefly the limit $z_1 \rightarrow \infty$. Then $f_o^e = f_M n_o^e$ and from the terms of next order in Eq. (9) there results an equation of the type

$$\frac{\partial f_2^e}{\partial \phi} = A_2(\phi) + C_2$$

and again $C_2 = 0$, $\partial f_2^e / \partial \phi = A_2$.

It is found that $j_2^{e\xi} \neq 0$ and is $O(\sigma^2 \tau)$. Also, $j_1^{ez} \neq 0$ and is $O(\sigma)$. If $L_2^z = \lambda R / \ell_e$, the continuity equation allows now a spreading such that at infinity (in this scale) $u_{\infty}^e = 0$ and $n^e (= n^i) = 1$, $\chi = 0$. Thus in this scale the probe perturbation can vanish at infinity.

IV. SOLUTION TO THE EQUATIONS

A. The Outer z Layer

In this section we shall be concerned with a proper solution of Eqs. (5), (9), and (10), as complete as necessary to determine the electron current to the probe to first order. (To zero order in σ , I^e vanishes.)

It was found in the last section that over distances of order $\lambda R/\ell_e$ and R , along and across B , respectively, the conditions at infinity could be satisfied. We define these "outer" variables:

$$z_2 = \frac{z \ell_e}{R \lambda} \quad , \quad \xi_r = \frac{\xi}{R} \quad .$$

Equations (5), (9), and (10) become

$$\mathcal{E}^2 \left[\xi_r^{-s} \frac{\partial}{\partial \xi_r} \xi_r^s \frac{\partial}{\partial \xi_r} + \sigma^2 \tau^2 \frac{\partial}{\partial z_2^2} \right] \chi = n^e - n^i \quad (15)$$

$$\begin{aligned} \frac{\partial f^e}{\partial \phi} + \sigma \cos \phi D_{\xi_r} f^e + \sigma^2 \tau D_{z_2} f^e - \\ - \sigma \sin \phi \left[\frac{sv}{\xi_r} + \frac{1}{v} \frac{\partial \chi}{\partial \xi_r} \right] \frac{\partial f^e}{\partial \phi} = \sigma \tau [c(f^e, f^e) + \\ + n^i P(f^e) + \mu P_\mu(f^e, f^i)] \end{aligned} \quad (16)$$

$$\begin{aligned} \sigma \left[v \xi \frac{\partial}{\partial \xi_r} - \beta^{-1} \frac{\partial \chi}{\partial \xi_r} \frac{\partial}{\partial v \xi} \right] f^i - \mu \frac{\partial f^i}{\partial \phi} - \sigma \frac{sv \eta}{\xi_r} \frac{\partial f^i}{\partial \phi} \\ + \sigma^2 \tau D_{z_2}^i f^i = \sigma \tau [c(f^i, f^i) + c(f^i, f^e)] \quad . \end{aligned} \quad (17)$$

Next, we introduce the expansion

$$f^e = f_o^e + \delta_1 f_1^e + \delta_2 f_2^e + \dots$$

$$f^i = f_o^i + \delta_1^i f_1^i + \dots$$

$$\chi = \chi_o + \delta_1^\chi \chi_1 + \dots$$

Then we obtain from Eq. (16),

$$\frac{\partial f_o^e}{\partial \phi} = 0$$

$$\left. \begin{aligned} \delta_1 \frac{\partial f_1^e}{\partial \phi} + \sigma \cos \phi D_{\xi_r} f_o^e &= 0 \\ c(f_o^e, f_o^e) + n_o^i P(f_o^e) &= 0 \end{aligned} \right\}$$

so that

$$\delta_1 f_1^e = -\sigma \sin \phi D_{\xi_r} f_o^e + \delta_{1c} f_{1c}^e$$

$$f_o^e = n_o^e f_M = n_o^e (2\pi)^{-3/2} \exp \left(-\frac{v^2}{2} \right)$$

(see App. B).

To following order,

$$\begin{aligned} & \delta_2 \frac{\partial f_2^e}{\partial \phi} + \sigma \delta_1 \cos \phi D_{\xi_r} f_1^e + \sigma \delta_1^\chi \cos \phi \frac{\partial \chi_1}{\partial \xi_r} \frac{\partial f_o^e}{\partial v_\perp} \\ & - \sigma \delta_1 \sin \phi \left(\frac{sv_\perp}{\xi_r} + \frac{1}{v_\perp} \frac{\partial \chi_o}{\partial \xi_r} \right) \frac{\partial f_1^e}{\partial \phi} \\ & = \sigma^2 \tau [c_L(f_o^e, -\sin \phi D_{\xi_r} f_o^e) + n_o^i P(-\sin \phi D_{\xi_r} f_o^e)] \end{aligned} \quad (1)$$

$$\begin{aligned} \sigma^2 \tau D_{z_2} f_o^e = \sigma \tau [\delta_{1c} \{ c_L(f_o^e, f_{1c}^e) + n_o^i P(f_{1c}^e) \} \\ + \delta_1^i n_1^i P(f_o^e) + \mu P_\mu(f_o^e, f_o^i)] \end{aligned} \quad (19b)$$

c_L is the result of linearizing $c(f_o^e + \delta_1 f_1^e, f_o^e + \delta_1 f_1^e)$. It will be seen below that f_o^i is Maxwellian. Then $\mu P_\mu = O(\mu^2)$ and should be removed from the last equation [for arbitrary β ; it is known in particular that the mean free path for energy exchange is $O(\lambda \mu^{-2})$].¹⁸ Of course, $\delta_1^i n_1^i P(f_o^e)$ vanishes.

There follows

$$\delta_{1c} = \sigma v_z f_o^e \frac{\partial}{\partial z_2} (\ln n_o^e - \chi_o) = c_L(f_o^e, f_{1c}^e) + n_o^i P(f_{1c}^e) \quad (20)$$

and j_1^{ez} can be obtained.

It is easy to integrate Eq. (19a) giving

$$\delta_2 f_2^e = \delta_{2c} f_{2c}^e + \delta_2 \int \frac{\partial f_2^e}{\partial \phi} d\phi \quad .$$

However, we shall need only $j_2^{e\xi}$; the only contribution comes from the right-hand side of Eq. (19a).

$$\delta_2 j_2^{e\xi} = \sigma^2 \tau \int d\bar{v} v_\eta [c_L(f_o^e, \sin \phi D_{\xi_r} f_o^e) + n_o^i P(\sin \phi D_{\xi_r} f_o^e)] \quad (21)$$

Then from the electron continuity equation a relation between n_o^e, n_o^i , and χ_o is obtained,

$$\sigma^3 \tau \left(\frac{\partial}{\partial z_2} j_1^{ez} + \xi_r^{-s} \frac{\partial}{\partial \xi_r} \xi_r^s j_2^{e\xi} \right) = 0 \quad (22)$$

From Eq. (15), $n_o^e = n_o^i$ and from Eq. (17),

$$\begin{aligned} \sigma \left[v_{\xi} \frac{\partial}{\partial \xi_r} - \beta^{-1} \frac{\partial \chi_o}{\partial \xi_r} \frac{\partial}{\partial v_{\xi}} \right] f_o^i - \mu \frac{\partial f_o^i}{\partial \phi} - \frac{\sigma s v_{\eta}}{\xi_r} \frac{\partial f_o^i}{\partial \phi} = \\ = \sigma \tau [c(f_o^i, f_o^i) + c(f_o^i, f_o^e)] \end{aligned}$$

Because of the form of f_o^e , $c(f_o^i, f_o^e) = O(\mu)$ and it should be dropped out of Eq. (19b). It is shown in App. B that the unique solution for f_o^i is then

$$f_o^i = A \exp(-\beta^{-1} \chi_o) (2\pi)^{-3/2} \exp \left[-\frac{v_{\perp}^2}{2} - \frac{(v_z - u_{oz}^{iz})^2}{2} \right] \quad (23)$$

where A and u_z^{io} are arbitrary functions of z_2 . Taking the limit $\xi_r \rightarrow \infty$ with fixed z_2 gives $A \equiv 1$, $u_o^{iz} \equiv 0$. Thus, while f_o^e is a local Maxwellian, f_o^i is a global Maxwellian distribution and $n_o^e = n_o^i = \exp(-\beta^{-1} \chi_o)$. If $\chi_o \approx 0$ in the $\xi_r - z_2$ plane, $n_o^e \approx 1$ and no diffusion would exist. Only in the limit $\beta \rightarrow 0$ is it possible to have $\chi_o \approx 0$ while $n_o^e \neq 1$.

Equation (22) involves only χ_o after these substitutions. If boundary conditions on $z_2 = 0$ are known $\chi_o = \chi_o(\xi_r, z_2)$ can be obtained. In general, a relation of the type $G(\chi_o, \partial \chi_o / \partial z_2) = 0$ is needed on $z_2 = 0$. It will be seen later that this relation involves σ ; and $\chi_o^* [= \chi_o(z_2 = 0, \xi_r < 1)]$ is slightly larger than $O(1)$ for $\chi_p = O(1)$. However, it is neither worth while nor easy in the partial differential equation to separate terms of almost equal order. Moreover, it is not clear that $\partial \chi_o^* / \partial z_2$ is not $O(1)$.

On the probe, $z = 0$, the expansion for f^e breaks down since all particles travel toward the probe there, and f_o^e cannot be isotropic. A boundary layer(s) will exist of the type classical in fluid mechanics where gradients are strong and the condition of perfect absorption by the probe can be satisfied. As

in fluid mechanics it is possible, however, that the "outer" problem is a complete one (except for the condition right at the body surface). To see this, consider $z_2 \rightarrow 0$ ($\xi_r < 1$). Assume that $\chi_o = O(\ln \Delta^{-\beta})$ there; then $n_o^e = O(\Delta)$, with Δ unknown yet. This results in $\delta_{lc} f_{lc}^e / f_o^e = O(\sigma/\Delta)$. Thus, $\sigma/\Delta < O(1)$ is a condition for our expansion; since Δ will be found to decrease with χ_p there is a maximum value of probe potential for which our procedure is valid. The results $n_o^e = n_o^i$ and $n_o^i = \exp(-\beta^{-1} \chi_o)$ are still correct.

Assuming $\sigma/\Delta < O(1)$ we consider the limit $\xi_r \rightarrow 1$. While over "most" (asymptotically) of the probe the characteristic length of ξ gradients should be R , it is possible that at $\xi_r = 1$ much larger gradients are needed. Far from the probe, where $n_o^e = O(1)$, this is not the case, but where $n_o^e = O(\Delta)$ (at small z_2 and $\xi_r < 1$), $j^{e\xi} = O(\sigma^2 \tau \Delta^2)$ while for $\xi_r > 1$, $j^{e\xi} = O(\sigma^2 \tau)$. The first term of Eq. (22) has to produce this change. Since $j^{ez} = O(\sigma)$ there results

$$\sigma \sigma^2 \tau = \sigma^2 \tau \Delta^2 / \delta \sigma / \delta$$

or $\Delta = \delta$, where $\delta = L^\xi / R$; $\sigma/\delta = \ell_e / L^\xi$, and $L^\xi \ll R$ is the characteristic length over which this fast change across B exists. $\delta_1 f_1^e / f_o^e = O(\sigma/\Delta)$ and $\sigma/\Delta < O(1)$ is again the condition for the validity of our expansion. However, only a Δ fraction of the probe area is subject to this large gradient and sensibly an error $O(\sigma)$ is produced in this case.

The result for f_o^i is still unmodified. As for Poisson's equation, the left-hand side is now $O(\xi^2 \Delta^{-2})$, and since the right-hand side is $O(\Delta)$ the error of quasineutrality is $O(\xi^2 / \Delta^3)$. If Δ is small enough so that

$\mathcal{E}^2/\Delta^3 = O(1)$, we should use instead of $n_o^e = n_o^i$,

$$\mathcal{E}^2 \xi_r^{-s} \frac{\partial}{\partial \xi_r} \xi_r^s \frac{\partial}{\partial \xi_r} \chi_o = n_o^e - \exp(-\beta^{-1} \chi_o) \quad . \quad (24)$$

This equation, together with Eq. (20), determines both χ_o and n_o^e . Some problem in finding boundary conditions for both could exist then, but this only over a region on the boundary of order $\mathcal{E}^{2/3}$ that for elliptic equations seems only important to $O(\mathcal{E}^{2/3})$. In the numerical results presented in Sec. V, $\mathcal{E}^2/\Delta^3 < O(1)$ will be assumed.

As soon as we find the relation $G[\chi_o, (\partial \chi_o / \partial z_2)] = 0$ on the line $z_2 = 0$, we have results for χ_o , n_o^e , and n_o^i on the whole $\xi_r - z_2$ space valid if σ/Δ and \mathcal{E}^2/Δ^3 [or $\mathcal{E}^{2/3}$ in case $\Delta^3 = O(\mathcal{E}^2)$] $< O(1)$. However, we have yet to determine which is the order of the terms neglected when writing $j^{ez} \approx j_1^{ez}$, $j^{e\xi} \approx j_2^{e\xi}$. If the terms of next order are taken in Eq. (16) it is possible to find that the order of both $\delta_2 j_2^{ez} / \delta_1 j_1^{ez}$, $\delta_3 j_3^{e\xi} / \delta_2 j_2^{e\xi}$ is the lowest of σ/Δ , δ_1^i/Δ , and δ_1^χ . δ_1^i/Δ can be seen to be $O(\mathcal{E}^2/\Delta^3)$ from Poisson's equation if quasineutrality is taken; in case Eq. (24) is adopted, it is of higher order. Thus, we come back to the same critical inequalities to be satisfied in our theory. As for δ_1^χ , analysis of Eq. (15) seems to rule out the possibility of its being larger than both σ/Δ and \mathcal{E}^2/Δ^3 .

It can be seen that collisions are local in the whole $\xi_r - z_2$ plane; hence, a collision operator other than F-P or B-L needs to be considered only if $\ell_e < \lambda_D$; that operator, which includes \underline{B} , is already known. (Where $n_o^e = O(\Delta)$ and at $\xi_r = 1$, $\lambda_D/\Delta^{1/2} < L^\xi = R\Delta$ if $\mathcal{E}^2 \approx \Delta^3$; but even if $\mathcal{E}^2 \approx \Delta^3$ collisions are local because $\ell_e < L^\xi$ and \underline{B} restricts to ℓ_e the impact parameter transverse to it.)

Then both j_1^{ez} and $j_2^{e\xi}$ can be found from Eqs. (20) and (21):

$$\frac{\beta + 1}{\beta} \frac{\partial \chi_o}{\partial z_2} \frac{\partial f_M}{\partial v_z} = c_L(f_M, f_{lc}^e) + P(f_{lc}^e)$$

$$j_2^{e\xi} = \frac{\beta + 1}{\beta} \exp\left(-\frac{2\chi_o}{\beta}\right) \frac{\partial \chi_o}{\partial \xi_r} \int dv v_\eta P\left(\frac{\partial f_M}{\partial v_\eta}\right) .$$

The term $c_L(f_M, \partial f_M / \partial v_\eta)$ does not contribute to $j_2^{e\xi}$ since the electrons do not gain average momentum from themselves (collisions are local). There result, in the present non-dimensional form,

$$j_1^{ez} = \frac{\gamma(Z_i)}{Z_i} \frac{\beta + 1}{\beta} \frac{2^{7/2}}{\pi^{1/2}} \frac{\ln \Lambda_\infty}{\ln \Lambda} \frac{\partial \chi_o}{\partial z_2} \quad (25a)$$

$$j_2^{e\xi} = Z_i \frac{\beta + 1}{\beta} \frac{2^{3/2}}{3\pi^{1/2}} \frac{\ln \Lambda}{\ln \Lambda_\infty} \exp\left(-\frac{2\chi_o}{\beta}\right) \frac{\partial \chi_o}{\partial \xi_r} . \quad (25b)$$

γ is the factor given by Spitzer¹⁹ to correct the Lorentz conductivity; $\ln \Lambda$ is spatially dependent since n_o^e is. When $\ell_e < \lambda_D$ no results are known to the author for j_1^{ez} (dc conductivity along \underline{B} when \underline{B} enters the collision process), although results exist for $j_1^{e\xi}$ for $\ell_e \ll \lambda_D$. Even if $\lambda_D < \ell_e$, the above simple expression for $\ln \Lambda$ may not be correct since the local Debye length and not λ_D should be compared.

In what follows we shall assume an average value $\langle \ln \Lambda \rangle$. This is basically because of the imperfect knowledge of transport coefficients in the presence of a strong magnetic field (some review is made in I). Also, the dominant effect is the exponential term in Eq. (25b) while the variation due to $\ln \Lambda$ can be negligible for large enough $\ln \Lambda_\infty$. Finally, the results will be seen below not to depend on $\langle \ln \Lambda \rangle$.

Using Eqs. (25a) and (25b) in Eq. (22) and defining

$$\psi = \frac{2\chi_0}{\beta}, \quad y = z_2 \frac{Z_i}{(12\gamma)^{1/2}} \frac{\langle \ln \Lambda \rangle}{\ln \Lambda_\infty}, \quad x = \xi_r$$

we obtain finally

$$\frac{\partial^2 \psi}{\partial y^2} + x^{-s} \frac{\partial}{\partial x} x^s e^{-\psi} \frac{\partial \psi}{\partial x} = 0 \quad (26)$$

$$\sigma j_1^{ez} = \sigma(\beta + 1) \left(\frac{8\gamma}{3\pi} \right)^{1/2} \frac{\partial \psi}{\partial y}. \quad (27)$$

This elliptic equation for ψ should be solved on the half-space $y \geq 0$ with the boundary conditions

$$\psi = 0 \quad \text{at infinity}$$

$$G(\psi, \frac{\partial \psi}{\partial y}) = 0 \quad \text{at } y = 0, \quad (28)$$

this last relation being yet unknown.

Before going to the analysis of the inner layers, some comments should be made. An important result is that the electric field is non-negligible very far from the probe; in fact, χ is not $O(\chi_p)$ and the field is not restricted to distances $O(\lambda_D)$ from the probe. Only in the case $\beta \rightarrow 0$ is χ small in this "outer" space.

Near the probe, χ has to decay to negative values or positive values of order unity; the large potential hill resulting will simplify the obtainment of a correct solution. However, strong restrictions are imposed by the large penetration of the potential. First, the plasma cannot be shorter (along \underline{B}) than $\lambda R / \ell_e$, otherwise the probe will have catastrophic effects

on the plasma. Fluctuations will set in and a steady state seems not possible; anomalous transport processes will appear. Second, even if the plasma is long enough, macroscopic variations should be very weak; otherwise the measurement would be non-local and the probe would interfere with the whole plasma.

If any of these conditions are violated it seems improbable that probes are of any use. Results would not be universal but would depend on the particular plasma, and spatial resolution could not be obtained. Thus, use of probes with strong magnetic field is strongly restricted. On the other hand, if these conditions are met, the present theory is a correct one in an asymptotic sense and results can easily be accurate to 10%. If poorer accuracy is not rejected the theory should be useful even if the above conditions are not quite satisfied. Moreover, in Sec. VI it will appear that results can be used for weaker fields down to the limit $B \rightarrow 0$; then the restrictions are weak too. Finally, a comparison could be made with the case $B = 0$ as far as the perturbation of the plasma is considered. In the case of the disc (the strip cannot be used) the current density, which is a measure of the perturbations, is $O(\sigma)$ at distances $O(\lambda R / \ell_e)$; in the case $B = 0$ for a cylinder which is frequently used, the current density is $O(\sigma \tau)$ at such distances. Thus, if $\tau = O(1)$ the comparison is not bad; moreover, at smaller and larger distances the cylinder disturbs more of the plasma, and transversely to \underline{B} the perturbation of our case decays as ξ^{-2} over distances of order of R .

B. The Inner z Layers

To satisfy the condition on the probe an inner variable is defined

$z_1 = z/L_1^z$ such that the term $D_{z_1} f^e$ is now comparable to the collision term.

Next we introduce the expansions

$$f^e = \Delta [g^e + \delta_1^* g_1^e + \dots]$$

$$\chi = \ln \Delta^{-\beta} + \chi_0^* + \delta_1^{\chi*} \chi_1^* + \dots$$

so that

$$\frac{\partial g_o^e}{\partial \phi} = 0$$

$$\delta_1^* \frac{\partial g_1^e}{\partial \phi} + \Delta^{-1} \sigma \cos \phi D_{\xi_r} \Delta g_o^e = 0 \quad (29)$$

$$\frac{\ell_e}{L_1^z} D_{z_1} g_o^e = \sigma \tau [\Delta c(g_o^e, g_o^e) + n_o^i P(g_o^e)] \quad (30)$$

Hence $L_1^z = \lambda \Delta^{-1}$; Poisson's equation still reduces to the quasineutrality

condition since $\lambda \Delta^{-1} \geq O(R)$. As for f_o^i our former result remains

obviously valid if $\lambda \Delta^{-1} > O(R)$; we shall comment later on the other case.

Here $\Delta = \lim_{z_2 \rightarrow 0} n_o^e(z_2)$ as obtained from the analysis of the z_2 layer and

thus depends on ξ_r . (We are presently concerned only with the range

$\xi_r < 1$.)

From Eq. (30) we obtain sensibly $n_o^e \sim e^{\chi_o^*}$; this behavior is incompatible with the knowledge we have that as $z_1 \rightarrow \infty$, $n_o^e \sim e^{-\chi_o^*/\beta}$ as the matching with the z_2 layer would require. The only possibility is that

$\chi_0^* \equiv 0$ so that in the inner region $\chi = \ln \Delta^{-\beta} + \delta_1^{\chi^*} \chi_1^* + \dots$. This result shows that the flattened part of the potential around the hill covers the z_1 layer, which basically will produce only a change in f^e as a function of \underline{v} . There has to be a new layer where a fast decay of χ to its value on the probe will occur. Gradients along \underline{B} will become large and electrons will experience free-streaming motion toward the probe. Before studying this region we shall take the limit $z_2 \rightarrow 0$ in our solution in the outer space, to obtain the matching condition at $z_1 \rightarrow \infty$; by rewriting it first in inner variables,

$$\begin{aligned} \chi &= \chi_0 \left(\frac{\sigma}{\Delta} z_1 \right) + \delta_1^{\chi} \chi_1 \left(\sigma / \Delta z_1 \right) + \dots \\ &= \chi_0(0) + \frac{\sigma}{\Delta} \left. \frac{\partial \chi_0}{\partial z_2} \right|_{z_2=0} z_1 + \delta_1^{\chi} \chi_1(0) + \dots \\ &= \ln \Delta^{-\beta} + \frac{\sigma}{\Delta} q z_1 + \delta_1^{\chi} \chi_1(0) + \dots \end{aligned} \quad (31)$$

where $q(\xi_r) \equiv \partial \chi_0 / \partial z_2 \big|_{z_2=0}$. Also

$$f^e = \Delta f_M - \sigma q \left\{ \frac{z_1}{\beta} f_M - \frac{\beta+1}{\beta} c^{-1} \frac{\partial f_M}{\partial v_z} \right\} + \sigma(\beta+1) \frac{\partial \Delta}{\partial \xi_r} \frac{\partial f_M}{\partial v_\eta} + \dots \quad (32)$$

where c^{-1} is such that

$$c_L(f_M, c^{-1} \frac{\partial f_M}{\partial v_z}) + P(c^{-1} \frac{\partial f_M}{\partial v_z}) = \frac{\partial f_M}{\partial v_z}.$$

In the z_0 region,

$$f^e = \tilde{\Delta} [\tilde{g}_0^e + \tilde{\delta}_1 \tilde{g}_1^e + \dots]$$

and there results

$$\begin{aligned}
 D_{z_0} \tilde{g}_0^e &= 0 \quad (33) \\
 \tilde{\delta}_1 \tilde{g}_1^e &= -\tilde{\Delta}^{-1} \sigma \sin \phi D_{\xi_r} \tilde{g}_0^e \tilde{\Delta} + \tilde{\delta}_{1c} \tilde{g}_{1c}^e \\
 \frac{\ell_e}{L_z^0} \left\{ \tilde{\delta}_{1c} D_{z_0} \tilde{g}_{1c}^e + \tilde{\delta}_1 \chi \frac{\partial \chi^1}{\partial z_0} \frac{\partial \tilde{g}_0^e}{\partial v_z} \right\} &= \tilde{\Delta} \sigma \tau [c(\tilde{g}_0^e, \tilde{g}_0^e) + \\
 &+ n_0^i / \tilde{\Delta} P(\tilde{g}_0^e)] .
 \end{aligned}$$

n_0^i should be much larger than $\tilde{\Delta}$ so that $\tilde{\delta}_{1c} = O(L_z^0 n_0^i / \lambda)$.

The solution for \tilde{g}_0^e is

$$\tilde{g}_0^e = g(v_z^2 - 2\tilde{\chi}_0) h(2^{1/2} [\chi_0 - \chi_p]^{1/2} - v_z) \quad (34)$$

where g is an arbitrary function and

$$\begin{aligned}
 h(x) &= 1 \quad x > 0 \\
 h(x) &= 0 \quad x < 0 .
 \end{aligned}$$

Because $dh/dx = \delta(x)$ (Dirac's delta function) and $x\delta(x) \equiv 0$, (34) satisfies identically Eq. (33) and the condition of perfect absorption on the probe, too.

We rewrite

$$g(v_z^2 - 2\tilde{\chi}_0) = e^{\tilde{\chi}_0} f_M \tilde{g}(v_z^2 - 2\chi_0, v_1^2)$$

to put in evidence some expected exponential behavior in g ; \tilde{g} is again arbitrary.

Equations (29) and (30) in z_1 should be solved now; the equation for \tilde{g}_0^e is nonlinear but we need not really solve it. It is enough to observe that the

z flux does not change in z_0 and z_1 ; the current at $z_0 = 0$ should then be made equal to the current at $z_2 = 0$. This will be the condition (28).

It is necessary, however, to determine $\tilde{\Delta}$ and \tilde{g} first. It should be noted that because of the large overshooting of χ , the discontinuity in the distribution function has moved to the positive high-velocity tail in v_z in the limit $z_0 \rightarrow \infty$; $\tilde{g}_0^e(z_1 \rightarrow \infty)$ is even in v_z to zero order. Also $\delta_1^{\chi*} \chi_1^* < O(1)$ and we would have in z_1 , splitting Eq. (30):

$$v_z \frac{\partial g_0^e}{\partial z_1} = c(g_0^e, g_0^e) + P(g_0^e) \quad (35)$$

$$\delta_{lc}^* v_z \frac{\partial g_{lc}^e}{\partial z_1} + \delta_1^{\chi*} \frac{\partial \chi_1^*}{\partial z_1} \frac{\partial g_0^e}{\partial v_z} = \delta_{lc}^* [c_L(g_0^e, g_{lc}^e) + P(g_{lc}^e)] + \frac{n_1^i}{\Delta} P(g_0^e) \quad (36)$$

g_0^e being even in v_z .

The equation for g_0^e is parabolic with z_1 acting as a time-like variable. Hence, if $g_0^e(z_1 = 0, v)$ is given, g_0^e is automatically obtained in z_1 . The matching condition at $z_1 \rightarrow 0$ gives not only $g_0^e(z_1 = 0)$ but the behavior as $z_1 \rightarrow 0$, in z_1 . However, it is easily found that the limiting form of f^e as $z_0 \rightarrow \infty$ satisfies Eqs. (29) and (30), as should be expected if the proper selection of layers has been made. Therefore only the function $g_0^e(z_1 = 0, v)$ is given really as independent data; i.e., $\tilde{\Delta}$ and the function \tilde{g} .

The point now is that because of the singular character of $g_0^e(z_1 = 0)$, that is, because of the h function, the splitting (35)-(36) cannot be made in the whole range of v_z for the whole z_1 layer. Around the discontinuity in

v_z the second term in the left-hand side of Eq. (30) is of the order of the first one and not small. Strong nonlinear effects are produced in this region by the collision term. The diffusion in velocity space that collisions produce does not allow the cut to persist, and a progressive smoothing will occur. In the limit $z_1 \rightarrow \infty$, therefore, that splitting will be valid.

The form for f^e as $z_1 \rightarrow \infty$ given by (32) has to come out automatically and is not a boundary condition. It can be observed immediately that if $g_o^e = f_M$, (35) is satisfied. From Eq. (31), as $z_1 \rightarrow \infty$,

$$\delta_1^* \frac{\partial \chi_1^*}{\partial z_1} \approx \frac{\sigma}{\Delta} q$$

and

$$\delta_{lc}^* g_{lc}^e = \left(-\frac{q f_M}{\beta} z_1 + \frac{\beta+1}{\beta} q c^{-1} \frac{\partial f_M}{\partial v_z} \right) \frac{\sigma}{\Delta}$$

is seen to satisfy (36). Therefore (32) is a solution of (29)-(30) in the limit $z_1 \rightarrow \infty$. This only means, of course, that no layer exists between z_1 and z_2 . There are other possible forms for $\delta_{lc}^* g_{lc}^e$ as $z_1 \rightarrow \infty$; in particular,

$$\delta_{lc}^* g_{lc}^e = q f_M z_1 \frac{\sigma}{\Delta} = (\delta_1^* \int \frac{\partial \chi_1^*}{\partial z_1} dz_1) f_M \quad (37)$$

$$\delta_{lc}^* g_{lc}^e = \frac{\sigma}{\Delta} q c^{-1} \frac{\partial f_M}{\partial v_z} \quad (38)$$

(The first one valid for an arbitrary χ_1^* .) However, it can be shown that these limiting forms are ruled out by Eq. (30) itself. Define

$$H = \int (g_o^e \ln g_o^e) v_z dv_z$$

There results

$$\frac{dH}{dz_1} = \int (\ln g_o^e + 1) [c(g_o^e, g_o^e) + P(g_o^e)] dv \leq 0 . \quad (39)$$

The inequality follows from the usual H-theorem for all known C's .

dH/dz_1 cannot vanish since from (30), too,

$$\frac{\partial}{\partial z_1} \int g_o^e v_z dv = 0 ,$$

and because $\int g_o^e v_z dv = \text{constant} \neq 0$ at $z_1 = 0$ the square bracket in (39) can never vanish. Thus, as $z_1 \rightarrow \infty$, g_{lc}^e has to depend on z_1 and to have a non-zero z flux, which rules out both (37) and (38).

Of course, one still has \tilde{g} as an arbitrary datum chosen such that as $z_1 \rightarrow \infty$ the desired behavior for f^e is obtained. Our problem is to find \tilde{g} ; in the end, to show that $\tilde{g} \equiv 1$. It seems that the difficulty is not to find a proper, unique \tilde{g} but that for any \tilde{g} , collisions will have made f^e Maxwellian to zero order in the limit $z_1 \rightarrow \infty$ and (32) can follow. To clarify and escape this difficulty, consider f^e given in (32) at $z_1 \rightarrow \infty$. We are going to use the overshooting in χ now, i.e., we shall take advantage of $\chi_p \ll \ln \Delta^{-\beta}$ or, better, $e^{\chi_p} \ll \Delta^{-\beta}$. From the value for Δ obtained below it can be seen that the condition $e^{\chi_p} \approx \Delta^{-\beta}$ is the same as $\sigma \approx \Delta$ and our former restriction $\sigma/\Delta < O(1)$ comes back again.

In the limit of an infinite overshooting ($\chi_p - \ln \Delta^{-\beta} \rightarrow -\infty$), the condition at $z_1 = 0$ is one of perfect reflection. From the argument in App. B, there follows that $g_o^e = \Delta f_M$ everywhere [since $\delta_1^{\chi^*} < O(1)$]; of course, $q = 0$ then. From continuity of the solution on the boundary conditions it is expected that for large but finite $(\chi_p - \ln \Delta^{-\beta})$ the change in boundary

condition at $z_1 = 0$ will affect to zero order only the positive v_z tail of the distribution function. Hence, it is possible to obtain immediately that

$$\tilde{g} \equiv 1, \quad \tilde{\Delta} \Delta^{-\beta} = \Delta.$$

The current arriving into the probe is

$$\begin{aligned} j^{ez} &\approx \Delta^{\beta+1} e^{\chi_p} \int f_M h(-v_z) v_z dv \\ &= - \Delta^{\beta+1} e^{\chi_p} (2\pi)^{-1/2}. \end{aligned} \quad (40)$$

Since $\Delta = \lim_{z_2 \rightarrow 0} e^{-\chi_o/\beta}$ we obtain for $\xi_r < 1$, from the equality of (27) and (40),

$$a e^{-b\psi} = - \frac{\partial \psi}{\partial y} \quad \text{at } z_2 = 0, \quad \xi_r < 1 \quad (41a)$$

where $b = (\beta + 1)/2$, $a = [\sigma (\beta + 1) e^{-\chi_p}]^{-1} (3/16\gamma)^{1/2}$. Equation (41a) can be written

$$\begin{aligned} \chi_o + \frac{\beta}{\beta+1} \ln \left(- \frac{\partial \chi_o}{\partial y} \right) &= \frac{\beta}{\beta+1} \chi_p + \ln \sigma^{-\beta/\beta+1} \\ &- \ln \left[\frac{\beta+1}{\beta} \left(\frac{64\gamma}{3} \right)^{1/2} \right]^{\beta/\beta+1} \quad \text{at } z_2 = 0 \end{aligned}$$

which shows that χ_o in z_2 is not $O(\chi_p)$ but $O(\ln \sigma^{-\beta/\beta+1})$. Also, the condition $\sigma/\Delta < O(1)$ becomes

$$\sigma^{\beta/\beta+1} e^{\chi_p/\beta+1} A^{-1/\beta+1} < O(1)$$

where $A = -(\partial \psi / \partial y)(\beta+1)(16\gamma/3)^{1/2}$. Because as Δ decreases A can become appreciably larger than unity, results should be accurate right up to $\chi_p = \ln \sigma^{-\beta}$.

In the range $\xi_r > 1$, the condition at $z_1 = 0$ is one of perfect reflection and the condition (28) results in

$$\frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad z_2 = 0, \quad \xi_r > 1. \quad (41b)$$

In I the density at $z_1 \rightarrow 0$ was matched to its value at $z_1 \rightarrow \infty$; because of this improper matching (the density is not conserved in z_1 to zero and first orders) an additional factor appeared in (41a) which limited more the valid range of χ_p and produced some anomalies in the probe characteristic (see Sec. V-A).

Some comments should be made now. First, it is obvious why the shape of the probe along $\underline{\underline{B}}$ has no sensible influence on the current as pointed out by Bohm.¹⁶ If its dimension along $\underline{\underline{B}}$ is, say, $O(R)$, the probe lies entirely in the $z_2 = 0$ plane; only its cross section appears in the formulation. This would not be the case if the probe were not perfectly absorbing.

Second, in the inner layers the problem is one-dimensional and Poisson's equation is unimportant, as suggested by Spivak¹² for the whole problem.

Third, both the decrease and the blurring in the $I-V_p$ diagram become clear. Because of the inhibition of transverse electron flux, any electron current is maintained over long distances along the field. Since the ions are motionless to $O(\mu)$ with respect to the electrons, any sensible flux would experience a friction over such a long distance that only a small electron current is possible.

Moreover, Poisson's equation imposes quasineutrality thus far from the probe; because ions are not inhibited in flowing across the field, only

a large electric potential can produce large differences in ion density, i. e., in electron density. Hence, an overshooting is built up inside the "shadow." The decay far from the probe of this potential hill draws the electron current.

As $\chi_p \rightarrow -\infty$, $I^e \rightarrow 0$ and the hill disappears; thus $\Delta \rightarrow 1$ and if $\tau = O(1)$ the result used for f_o^i is not valid in z_1 . But then the whole z_1 layer collapses. At $\chi_p = 0$ no kink in the $I-V_p$ diagram should be expected because for the inner layers the effective plasma potential is given by the overshooting. As χ_p increases this will disappear, as shown in Sec. V, and the present theory breaks down.

Finally, although difficult of proof without a much more detailed analysis, it seems that the error involved in stating the equality of total current to the probe to that reaching the plane $z_2 = 0$ is $O(\sigma)$. Also, when $\Delta < O(1)$, the ion current is expected to decrease from its value for $B = 0$ at the same χ_p .

V. THE PROBE CHARACTERISTIC

A. Analysis of the $I-V_p$ Diagram

We proceed now to discuss the determination of essential parts of the $I-V_p$ diagram. From now on, whenever the ion current, I^i , is taken into account, a cylindrical or spherical probe will be assumed; also $\ell_i > R$.

The floating potential, $\chi_p = \chi_f$, where I vanishes, is the most accessible result from experiments; moreover, the plasma is disturbed the least. Therefore an accurate theoretical determination of χ_f is important. Since the plasma potential is unknown, a check between both

results relates the plasma potential to the other unknowns of our problem.

To find χ_f we write

$$I^e(\chi_f) = I^i(\chi_f) \quad .$$

For large, negative χ_p , the effect of B on both I^e and I^i becomes small. We shall find then that there is a small change in I^e for large $|\chi_f|$ from the value for $B = 0$; and a much smaller change in I^i , for which we can then use results by other authors for $B = 0$. The change in χ_f is small but non-negligible.

Let us consider first I^e . If in Eq. (41a) we let $\chi_p \rightarrow -\infty$, there results $\psi(x < 1, 0) \rightarrow 0$. We can expand $\psi = \psi_0 + \psi_1 + \dots$ and obtain linear equations for ψ_j , $j = 0, 1, \dots$. For ψ_0 we get

$$\frac{\partial^2 \psi_0}{\partial y^2} + x^{-s} \frac{\partial}{\partial x} x^s \frac{\partial \psi_0}{\partial x} = 0$$

with the boundary conditions

$$\begin{aligned} \psi_0 &= 0 && \text{at infinity} \\ \frac{\partial \psi_0}{\partial y} &= 0 && \text{at } y = 0, x > 1 \\ \frac{\partial \psi_0}{\partial y} &= -a && \text{at } y = 0, x < 1. \end{aligned}$$

As soon as we know $\psi_0(x < 1, y = 0)$ we obtain from the left-hand side of Eq. (41a),

$$I^e \approx 2 e N_\infty \left(\frac{kT_\infty}{m_e} \right)^{1/2} R^2 2\pi \frac{\exp \chi_p}{(2\pi)^{1/2}} \int_0^1 x dx \left(1 - \frac{\beta+1}{2} \psi_0 \right).$$

It is obvious that the two-dimensional problem is not well posed since Laplace's equation has a logarithmic divergence at infinity. From now on we shall take $s = 1$.

To solve for ψ_0 we use an integral transform approach; we obtain

$$\psi_0(x, y) = \int_0^\infty \tilde{\psi}(q) e^{-qy} J_0(xq) dq.$$

The conditions at $y = 0$ give

$$\begin{aligned} \int_0^\infty \tilde{\psi}(q) q J_0(xq) dq &= a & x < 1 \\ &= 0 & x > 1 \end{aligned}$$

By inverting the Hankel transform,

$$\tilde{\psi}(q) = a q^{-1} J_1(q)$$

and

$$\begin{aligned} a^{-1} \psi_0(x, y) &= \frac{2x^{1/2}}{\pi k} E(k) + \frac{(1-x^2) k K(k)}{2\pi x^{1/2}} + \frac{y \Lambda_0(\beta, \alpha)}{2} - y & x < 1 \\ &= \frac{2E(k)}{\pi k} - y/2 & x = 1 \\ &= \frac{2x^{1/2} E(k)}{\pi k} + \frac{(1-x^2) k K(k)}{2\pi x^{1/2}} - \frac{y \Lambda_0(\beta, \alpha)}{2} & x > 1 \\ k = \sin \alpha &= \frac{2x^{1/2}}{[y^2 + (x+1)^2]^{1/2}}, \quad \sin \beta = y[y^2 + (1-x)^2]^{-1/2} \end{aligned}$$

and K , E , and Λ_0 are the complete and Heuman's elliptic integrals.

For $y = 0$, $x < 1$, $\psi_0 = 2\pi^{-1} a E(x)$ and thus

$$I^e = e N_\infty \left(\frac{kT_\infty}{m_e} \right)^{1/2} 2\pi R^2 \frac{e^{\chi_p}}{(2\pi)^{1/2}} \left[1 - \frac{e^{\chi_p}}{(3\gamma)^{1/2} \pi \sigma} \right]. \quad (42)$$

Let us now consider I^i . The point to be made is that if $\mathcal{E} < O(1)$ and χ_p is negative enough the ion current is somehow insensitive to certain modifications. Assume first $\tau < O(1)$. We refer the reader to Figs. 20, 21, 27b, and 39 in Ref. 5. It can be inferred that (if roughly $\mathcal{E} < 0.1$ and $\chi_p < -3.5$) for changes of order unity in χ_p (and even in the form of f^i at infinity), $\Delta I^i/I^i = O(10^{-1})$; for changes in the geometry of the probe, again I^i is modified only slightly and it does not change at all by assuming perfectly repelled electrons ($n^e = e^\chi$).

At $\chi_p \approx \chi_f$ we found a small correction for I^e with respect to the value for $B = 0$. The correction is connected with a small overshooting of χ (amounting to a decrease in χ_p) and a distortion on the spherical symmetry (amounting to a change in shape of the probe). Also, n^e will not have the same spatial dependence as for $B = 0$; but $n^e \approx e^\chi$. Therefore, all these small effects will produce higher-order modifications in I^i as deduced from the considerations of the preceding paragraph. We can use the results for I^i from Ref. 5.

In the case $\tau = O(1)$, the information from Ref. 9 is very incomplete. However, in Fig. 11 of Ref. 9 it can be seen that under the same conditions for \mathcal{E} and χ_p , $dI^i/d\chi_p$ is very small. Although it is not so clear for the effect of geometrical distortion, it is quite probable that again assuming $n^e = e^\chi$ will not modify I^i . Thus the results for the case $B = 0$ can be used for I^i . Those from Ref. 9 are not complete and are quantitatively valid only for weakly ionized gases.

It should be pointed out that as β decreases the above statements on I^i become less true. Thus, the conditions on β, τ , and ξ , already taken advantage of in Sec. IV, have also this motivation of allowing the use of results for I^i obtained with $B = 0$.

A second feature of the characteristic which is immediately obtained experimentally is the slope at zero current. Again we can use for dI^i/dV_p the values for $B = 0$. In fact, $d^2 \ln I^i / d\chi_p^2 \ll d \ln I^i / d\chi_p$ and the approximation is still better. As for dI^e/dV_p , we obtain it from Eq. (42).

For $\chi_p < -\chi_f$ all the arguments about I^i given above remain valid. We have both components of the current for χ_p up to $-\infty$. In particular, the behavior of I as $\chi_p \rightarrow -\infty$ can be taken from the limit $B \rightarrow 0$.

When $\chi_p \gtrsim -3$, I^i decreases rapidly and the decrease depends strongly on the particular conditions. However, I^e increases then, making the ratio I^e/I^i large. To solve for I^e , numerical computations are necessary, but a certain amount of information can be obtained analytically.

Let us assume that the boundary condition for $x < 1$, $y = 0$ were $\psi = \alpha$ where α is a given constant. $\partial\psi(x < 1, y = 0)/\partial y$ would depend on α and on x . We can talk, however, in some average sense and neglect the x dependence. Of course what follows has only an approximate, but meaningful, value. Then for the current electron density on the probe we have

$$j^e = \frac{e\chi_p e^{-b\alpha}}{(2\pi)^{1/2}} = \sigma^* b\Phi(\alpha) \quad (43)$$

where $\sigma^* = (32\gamma/3\pi)^{1/2} \sigma$ and $\Phi(\alpha) = -\partial\psi/\partial y$ at $x < 1, y = 0$. $\Phi(\alpha)$ depends only on the differential equation for ψ ; if this were linear $\Phi(\alpha) = E\alpha$ with E a constant. For small α this is certainly true. For all α , $d\Phi/d\alpha \geq 0$.

We obtain next an implicit equation for j^e :

$$j^e = \sigma^* b \Phi \left(\frac{\chi_p - \ln j^e - \ln(2\pi)^{1/2}}{b} \right) .$$

For $\chi_p \rightarrow -\infty$, Φ is linear; thus

$$j^e = \sigma^* E [\chi_p - \ln j^e - \ln(2\pi)^{1/2}] .$$

The dependence on b disappears and for $|\chi_p|$ large enough, $j^e \sim e^{\chi_p/(2\pi)^{1/2}}$.

If Φ were linear for all α it would come out for $\chi_p \rightarrow +\infty$,

$$j^e + \sigma^* E \ln j^e = \sigma^* E \chi_p + ct \quad (44)$$

or

$$j^e \sim \sigma^* E (\chi_p - \ln \chi_p) ,$$

with $\sigma^* E = O(\sigma)$ as a limiting slope. However, we can show that this result is not valid. In effect,

$$\chi_p \sim e^{b\alpha \Phi(\alpha)}$$

so that $d\alpha/d\chi_p > 0$. Then $d\Phi/d\chi_p > 0$ or

$$\frac{d}{d\chi_p} (\chi_p - b\alpha) > 0 ,$$

i.e.,

$$\frac{d\chi^*}{d\chi_p} < \beta/\beta_{+1} < 1$$

where $\chi^* = \chi(x < 1, y = 0)$ and represents the overshooting. Hence, the overshooting tends to disappear or, what is the same, $\Delta \rightarrow O(\sigma)$ and the formulation of the last section breaks down. Because of the exponential character of the left-hand side of Eq. (43),

$$\frac{d\chi^*}{d\chi_p} \approx \frac{\beta}{\beta+1}$$

and for $\beta \rightarrow \infty$ (cold electrons or formal limit $Z_i \rightarrow 0$) the overshooting persists. For small β it does not change; we saw it was negligible.

Therefore the limit $\chi_p \rightarrow +\infty$ for fixed σ cannot be studied with the present formulation.

Moreover, it seems that $d\Phi/d\alpha$ should decay as α grows because of the exponential factor in the second term of Eq. (26). The slope $dj^e/d\chi_p$ would fall fast as χ_p grows and it is possible that a zero slope is practically reached before the overshooting has disappeared; j^e will depend now on b and $dj^e/db > 0$. We give below the dependence of j^e on σ^* , χ_p , and b :

$$\frac{\partial j^e}{\partial \sigma^*} = \frac{j^e}{j^e + \frac{d\Phi}{d\alpha} \sigma^*} \frac{j^e}{\sigma^*} > 0 \quad (45a)$$

$$\frac{\partial j^e}{\partial \chi_p} = \frac{j^e}{j^e + \frac{d\Phi}{d\alpha} \sigma^*} \frac{d\Phi}{d\alpha} \sigma^* > 0 \quad (45b)$$

$$\frac{\partial j^e}{\partial b} = \frac{j^e}{j^e + \frac{d\Phi}{d\alpha} \sigma^*} \sigma^* \left(\Phi - \frac{d\Phi}{d\alpha} \alpha \right) . \quad (45c)$$

B. Numerical Results

Equation (26), together with the boundary conditions (41a,b), was solved using the University of Colorado CDC 3600 computer. The behavior of ψ at infinity was obtained analytically using an asymptotic coordinate expansion for large $\rho = (x^2 + y^2)^{1/2}$. Thus the domain of integration was finite. For large ρ , ψ is small, and, expanding $\psi = \sum_k \psi_k$ where $\psi_k/\psi_{k-1} \rightarrow 0$ as $\rho \rightarrow \infty$, we get

$$\nabla \cdot \nabla \psi_k = \Theta_k(\psi_0, \dots, \psi_{k-1}) .$$

For the homogeneous part, this results in

$$\psi_H = \sum_n P_n(\cos\theta) \left[\frac{a_n}{\rho^{n+1}} + b_n \rho^n \right]$$

where P_n are Legendre polynomials and $y = \rho \cos\theta$. Obviously $b_n = 0$ for all n and also $a_n = 0$ for odd n , because $y = 0$ is a plane of symmetry.

Thus

$$\psi_H = \frac{a_0}{\rho} P_0 + \frac{a_2}{\rho^3} P_2 + \dots .$$

We obtain next for ψ_1 ,

$$\nabla \cdot \nabla \psi_1 = \Theta_1(\psi_0) = \frac{1}{x} \frac{\partial}{\partial x} x \frac{a_0}{\rho} \frac{\partial}{\partial x} \frac{a_0}{\rho} .$$

Using spherical coordinates in both sides of this equation and expanding Θ_1 in Legendre polynomials results in

$$\psi_1 = \frac{a_0^2 \cos^2 \theta}{\rho^2} .$$

Hence for large ρ ,

$$\psi = \frac{a_0}{\rho} + \frac{a_0^2 \cos^2 \theta}{\rho^2} + O(\rho^{-3})$$

A rectangular area was used as the domain of integration, $x = 0$ and $y = 0$ being two sides. On the other two, the last equation was taken as a boundary condition, eliminating a_0 between ψ and $\nabla\psi$.

Although no use was made of it, a relation between the behavior of ψ on separate regions of the domain was available as a criterion of convergence; an over-relaxation factor could be obtained from it. To derive that relation we rewrite Eq. (26) as

$$\nabla \cdot \nabla \psi = \frac{1}{x} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} (e^{-\psi} + \psi) \equiv \Theta$$

This is a Poisson-like equation; after integrating over the whole space, the right-hand side vanishes. Hence

$$\int_V \nabla \cdot \nabla \psi dV \equiv \int_S \nabla \psi \cdot \underline{dS} = 0$$

from which

$$\int_0^1 x dx \left. \frac{\partial \psi}{\partial y} \right|_{y=0} + a_0 = 0$$

Since the spatial "charge," Θ , gives a zero total result, a_0 , which represents the total charge in a solution to the Poisson equation, must be equal to the charge on the disc.

Equation (26) was changed into a finite difference equation and solved by row-relaxation. To have an initial estimate for ψ we began with a small

value of a , for which the solution was found analytically above. An appropriate scaling was used as initialization for the next larger value of a .

In Fig. 1 the nondimensional current $I^e (m_e/kT_\infty)^{1/2} (e N_\infty R^2)^{-1}$ is given as a function of χ_p for $\beta = 1$ and several σ . Only values of $\chi_p \leq 0$ are given and the variation of I with β is not included. When the computations were made it was unfortunate that an improper matching was used and a factor $2[1 + \operatorname{erf}(\beta\psi/2 - \chi_p)^{1/2}]^{-1}$ appeared in the left-hand side of Eq. (41a). It produced two spurious effects: first, it gave $\partial I/\partial b < 0$ as presented in I, contrary to what is expected from (45c). This is because if the above factor is included the denominator of (45c) is still positive but a new and strong negative term adds to $[\Phi - (d\Phi/d\alpha)\alpha]$. Second, $I(\chi_p)$ turned upwards when χ_p was approaching $\chi^* = \beta\alpha/2$. The reason is that the error function causes $\partial I/\partial \chi_p$ to grow to infinity for $\chi_p \sim \beta\alpha/2$; $[d/dt \operatorname{erf} t^{1/2} = e^{-t}/2t^{1/2}]$. Moreover, the spurious factor restricted the range of validity of the theory since it can be seen that the overshooting disappears for large χ_p when that is not present. Some problem of storage was found also in the 3600 computer for $\chi_p \gtrsim 2$. It is intended to present in the future more complete numerical results, extended to positive values of χ_p that would allow determination of the character of $d\Phi/d\alpha$ for large α , and give the variation with β . Some extensions discussed in the next section will be included.

VI. DISCUSSION

A. Extensions of the Theory

A number of effects can be included in the present theory in a straightforward way.

Non-uniform work functions were considered by Medicus²⁰ and are easily allowed for here if cylindrical symmetry is preserved. (Uniform work functions are taken into account by shifting the origin of potentials.)

Second, non-perfectly absorbing probes can be included in the theory. Because of the well-defined motion of the electrons toward the probe, this amounts again to a modification of Eq. (41a) where x will appear explicitly. Here the shape of the probe along B will be of importance. The reflection coefficient should not be near 1.

Third, recombination is easily allowed for; in the $z_2 - \xi_r$ space, $n^e \sim n^i \sim e^{-\psi/2}$ and the continuity equation (26) would require only a term $\omega e^{-\psi}$ where ω would be a non-dimensional recombination coefficient. In the inner z layers, gradients are stronger, densities weaker, so that recombination can be neglected.

In this case we can treat the cylindrical problem ($s = 0$). Even if $\omega = 0$ it seems simple to obtain the results for an elongated ellipsoidal probe.

Two major extensions are the allowance for anomalous diffusion and weakly ionized gases. The first does not seem simple even if a rational theory existed for the transport processes in that case; this is because the inner layers would be difficult to analyze. We doubt, however, whether

such a theory would be useful here because the measurement would probably be non-local. As for weakly ionized gases, no difficulties seem to appear. They will be considered in future work.

As for the strength of \underline{B} the formulation of Sec. IV required only $\mu/\sigma \leq O(1)$ (which, of course, allows $\ell_i < R$ but not $\ell_i \ll R$). The limit $\sigma \rightarrow 0$ has a proper behavior in our formulation but $\mu \rightarrow 0$ at the same time. In Sec. V, when I^i was considered, the more restrictive $\ell_i > R$ was needed. Because of $\mathcal{E} < O(1)$, $\ell_i \gg R$ is not required.

To allow for $\mathcal{E} \sim 1$ has the additional, but perhaps solvable, complication of lacking some boundary condition in the $z_2 - \xi_r$ space for n_o^e or χ_o .

The present theory required $\sigma < O(1)$. However, in the limit $\sigma \rightarrow \infty$ our solution goes back to the known result

$$j^e = e \chi_p / (2\pi)^{1/2}$$

when only negative χ_p are considered. Thus it seems that the present solution is valid for weaker magnetic fields down to the limit $B = 0$ for values of χ_p up to that for which $\chi_p \sim \chi^*$ (for $B = 0$, $\chi_p \sim 0$). The overshooting thus has the property of extending to positive χ_p the range for which the simple result for a repelled species is valid. [One should take a plane probe (a disc for $s = 1$) for such small B 's ; results for I^i exist then only in an approximate way.]

B. Conclusions

The present study is a consistent asymptotic analysis in the limit of some small parameters approaching zero. The order of neglected terms is given. The dependence on χ_p , σ , β , Z_1 (and μ for the part of the diagram where I^1 is important) appears explicitly. Although τ determines the extent of the perturbation by the probe along \underline{B} , it does not enter into the result. The physical reason for this is that while the collecting fields inside the probe "shadow" decrease as τ decreases, the extension of the perturbation is larger and transverse diffusion fills the shadow over longer distances. As for \mathcal{E} , the current reaches a finite value in the limit $\mathcal{E} \rightarrow 0$ (contrary to the limit $\sigma \rightarrow 0$ for which $I^e \rightarrow 0$). Both $\mathcal{E}\tau$ and $\mathcal{E}\sigma^{-1}$ would enter into the solution if the local $\ln \Lambda$ were used instead of an average.

Because of this average the accuracy does not exceed $O[(\ln \Lambda)^{-1}] = O(10^{-1})$. While transport coefficients are not readily available for all cases, this is an extrinsic difficulty and some improvement can be reached easily. As for the accuracy of the asymptotic expansion, it seems that the critical errors are $O(\sigma/\Delta)$ and $O(\mathcal{E}^2/\Delta^3)$. If $\mathcal{E}^2/\Delta^3 = O(1)$, use of Poisson's equation in a narrow region makes again valid the solution with an error $O(\mathcal{E}^{2/3})$. Because $\Delta = O(\sigma^{1/\beta+1} e^{\chi_p/\beta+1})$, an upper limit to χ_p is imposed by the theory. It seems that results are good right up to χ_p such that $\Delta = \sigma$. The errors in the numerical computations are $O(10^{-2})$.

The most important feature of the solution is the existence of a large potential hill inside the "shadow." This made the solution easy and clarified

its validity for weak magnetic fields. The overshooting is perhaps of interest in other situations where plasma, field, and body interact. The possibility of using the results for arbitrary B is very important in practice.

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APPENDIX A

For a typical thermally ionized cesium plasma,

$$T_{\infty} \approx 2.3 \times 10^3 \text{ } ^\circ\text{K}$$

$$\beta \approx 1, \quad Z_1 = 1, \quad \mu \approx 0.002$$

$$10^{11} < N_{\infty} < 10^{13} \text{ (cm}^{-3}\text{)};$$

also

$$10^3 < B < 3 \times 10^4 \text{ (gauss)}$$

$$10^{-1} > R > 10^{-3} \text{ (cm)}.$$

The ranges of variation for ℓ_e , λ_D , R , λ , and ℓ_i are shown (in cm) in Fig. 2.

APPENDIX B

The equation for f^e ,

$$0 = c(f^e, f^e) + n^i P(f^e) \quad (B-1)$$

has as unique solution an isotropic Maxwellian distribution. While physically apparent, it can be proved simply. Define $H = \int d\bar{v} f^e \ln f^e$ and introduce $\partial f^e / \partial t$ in the left-hand side of (B-1). Then

$$\begin{aligned} \frac{dH}{dt} &= \int d\bar{v} (\ln f^e + 1) c(f^e, f^e) + n^i \int d\bar{v} (\ln f^e + 1) P(f^e) \\ &\equiv \frac{dH_1}{dt} + \frac{dH_2}{dt} . \end{aligned}$$

The usual H-theorem gives $dH_1/dt \leq 0$. Since

$$P(f^e) = \frac{A}{v^3} \frac{\partial}{\partial v} (1 - v^2) \frac{\partial f^e}{\partial v}$$

($A > 0$, $v = v_z/v$, $\partial f^e / \partial \phi = 0$) there results

$$\begin{aligned} \frac{dH_2}{dt} &= A n^i 2\pi \int_0^\infty \frac{dv}{v} \int_{-1}^1 dv (\ln f^e + 1) \frac{\partial}{\partial v} (1 - v^2) \frac{\partial f^e}{\partial v} \\ &= -2\pi A n^i \int_0^\infty \frac{dv}{v} \int_{-1}^1 dv (1 - v^2) \left(\frac{\partial f^e}{\partial v} \right)^2 \frac{1}{f^e} \leq 0 . \end{aligned}$$

Both dH_1/dt and dH_2/dt have to vanish independently for (B-1) to be satisfied; thus, f^e is Maxwellian and isotropic, $\partial f^e / \partial v = 0$. If $\partial f^e / \partial \phi = 0$ is not assumed, the conclusion follows anyhow.

The proof of Eq. (23) is the one used for a non-uniform gas in a vessel of perfectly reflecting walls and under the action of a potential field.²¹ However, no integration is made over z_2 : define $H = \int d\xi_r \int d\underline{v} f^i \ln f^i$ (for $s = 1$ Cartesian coordinates should be used too, and the integration should be made over both spatial coordinates perpendicular to \underline{B}). The limits for the ξ_r integral can be taken far enough so that there $f(v_\xi) = f(-v_\xi)$. Then the argument in Ref. 20 is reproduced. Only an arbitrary z_2 dependence is left as a factor of f^i and in u^{iz} which can be non-vanishing.

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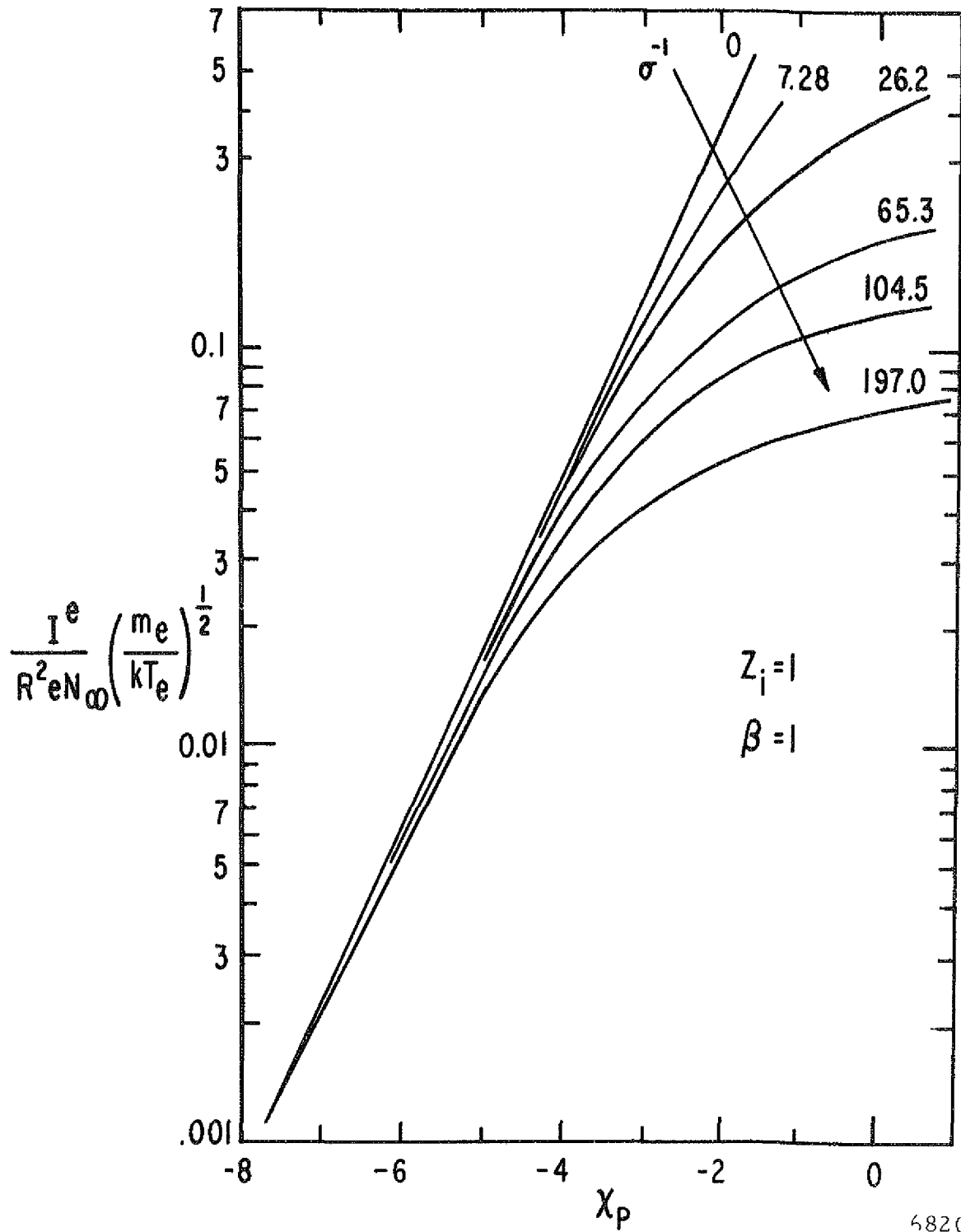
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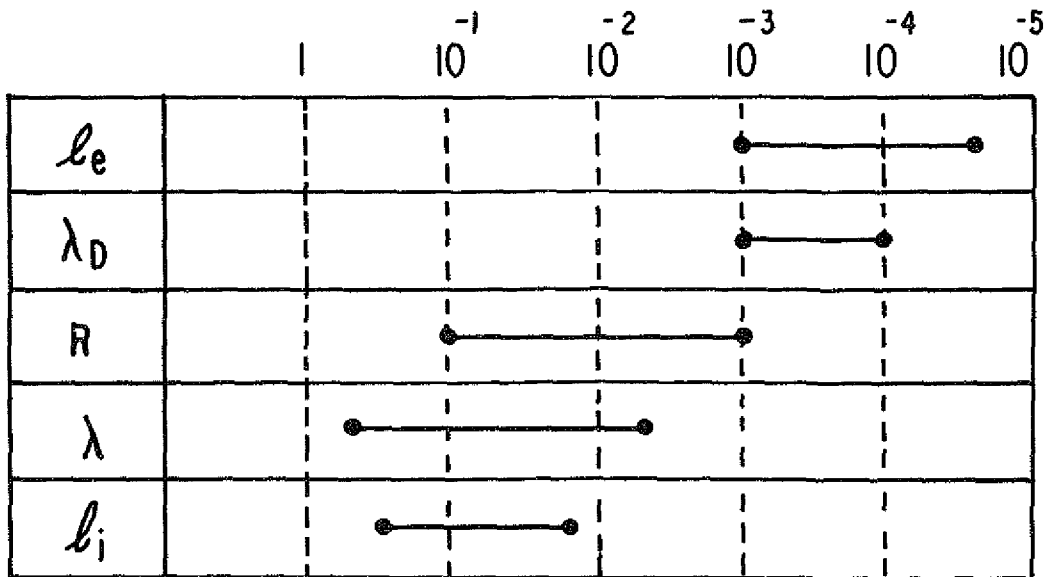
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²¹S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge University Press, 1960), Chap. 4.



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Fig. 1. The electron current as a function of χ_p and σ .



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Fig. 2. Typical ranges of characteristic lengths in a Cs plasma.

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